Gridline Indifference Graphs

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Abstract

Indifference graphs can be realized on a line with vertices adjacent whenever they are within a given distance. These well-studied graphs have applications to many fields including ecology, cluster theory, and psychology, in the placement of objects in a single dimension. The extension to the grid and higher dimensions has been considered in e.g. Goodman's study of perception (1977); we introduce *gridline indifference graphs*, which can be realized in the plane with vertices adjacent whenever they are within a given distance and on a common vertical or horizontal line. We obtain full and partial characterizations, under a natural restriction, in terms of forbidden subgraphs, extreme points, and tree-clique graphs. These graphs are extended to higher dimensions.

Keywords: Indifference graph; Tree-clique graph

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1 Introduction

An *indifference* (or *unit interval* or *proper interval*) *graph* is one that can be realized on the line with vertices adjacent whenever they are at most some fixed distance apart. There are applications of indifference graphs in many areas including biology, communication, economics, psychology, archaeology, transportation, and ecology (see Roberts (1976, 1978)). Another area of application is perception. Goodman (1977) studies predicates that can be used to order stimuli (e.g. objects, sounds). Roberts (1968) shows that some of the predicates studied by Goodman lead precisely to indifference graphs. Roberts (1973) discusses how visual perception in a static view can be modeled by indifference graphs. Stimuli that are close in some sense are indistinguishable. A natural extension is where there are $p \geq 2$ types of objects to be ordered, where the stimuli types are independent. Such stimuli types might include objects, colors, and light intensity. Goodman (1977) considers the extension to two dimensions natural and applies some of his predicates to the grid. These ideas suggest extending the indifference graphs to two and higher dimensional grids. This gives us an extension of indifference graphs that we call *gridline indifference graphs*.

Still another motivation for studying gridline indifference graphs is *cluster theory*. Often one wishes to aggregate data into groups that share some common property. If each datum corresponds to a vertex and edges are between data sharing the property, then we seek a group of pairwise adjacent vertices. When vertices are along a line and such groups correspond to intervals of fixed length then we obtain an indifference graph (Roberts, 1978). If clustered vertices are grouped in the same way along a tree structure rather than a line, then we obtain a *tree-clique graph*. These graphs were introduced by Batbedat (1990) (who called them *arba graphs*), and were studied by Gutierrez and Oubiña (1996), who identified several metric characterizations.

Before stating three important characterizations of indifference graphs, we present some graph theory definitions. We follow the terminology of Bondy and Murty (1976); any undefined terms can be found there. A graph is a pair $G = (V, E)$ where V (or $V(G)$) is the vertex set and the edge set E (or $E(G)$) is a subset of $\binom{V}{2} := \{uv(= vu)$: u and v are distinct vertices$\}$. All graphs are finite and *simple*, that is, they have no multiple edges or loops, and are undirected. With an abuse of language, and when no confusion is possible, we often refer to a vertex or edge as being in a graph G, and write for example $v \in G$ or $uw \in G$ instead of $v \in V(G)$ or $uw \in E(G)$.

An *indifference graph* is a graph that can be realized on the line such that two vertices are adjacent whenever the distance between them is at most ϵ, where $\epsilon \in \mathbb{R}^+$ is a fixed number.

A *(partial) subgraph* G' of a graph $G = (V, E)$ is a pair (V', E') where $V' \subseteq V$ and $E' \subseteq \binom{V'}{2} \cap E$, i.e. E' is a subset of the edges in E restricted to V'. An *induced* subgraph G' of G is a subgraph where $E' = \binom{V'}{2} \cap E$, i.e. E' is all of the edges of E restricted to V'. If G and H are graphs, then G is H*-free* means no induced subgraph of G is of type (isomorphic to) H. A *triangulated* graph is a graph having no *hole* – an induced cycle of length at least four.

Figure 1.1 shows a G_0, a *net*, and a *claw* (also called a $K_{1,3}$). (A net is in fact the *complement* of G_0, that is, it is obtained from G_0 by removing each edge and including every other possible edge.)

The following theorem characterizes indifference graphs in terms of forbidden subgraphs. We obtain the analogous result for higher dimension indifference graphs in section 3.

Theorem 1.1 (Roberts, 1969): Suppose G is a graph. Then

$$G \text{ is an indifference graph} \iff \begin{cases} \text{(a)} & G \text{ is triangulated} \\ \text{(b)} & G \text{ is } G_0\text{-, net-, and claw-free} \end{cases} \qquad \blacksquare$$

A *clique* is a set of vertices that induce a *complete* subgraph – a graph in which all vertices are adjacent – and which is maximal (with respect to set inclusion). Sometimes a clique refers to the complete subgraph induced by these vertices; which meaning of clique is intended will be clear from the context or will be specifically stated. An *extreme point* is a vertex a that is *simplicial*, that is, is in only one clique, and, if $ax, ay \in G$ where x and y are each in some clique other than the one containing a, then there is a vertex z such that $xz, yz \in G$ and $az \notin G$. The *closed neighborhood* of a vertex v, denoted $N[v]$, is the vertex set consisting of v and all of its neighbors. The *reduced graph* of graph G, denoted G^*, is defined as follows: Let $\mathbf{R}$ be the equivalence relation on $V(G)$ defined by $x\mathbf{R}y$ whenever $N[x] = N[y]$. Denote by $[x]$ the equivalence class containing x. Define G^* by taking the vertex set to be the equivalence classes and take $[x][y] \in E(G^*)$ whenever $xy \in E(G)$. We say that G^* is *reduced* (or *canonical*), and note that no two vertices of G^* have the same closed neighborhood.

The following theorem characterizes indifference graphs in terms of extreme points in the reduced graph. We obtain a partial analog for higher dimension indifference graphs in section 5.

Theorem 1.2 (Roberts, 1969): Suppose G is a graph. Then

$$G \text{ is an indifference graph} \Longleftrightarrow \begin{cases} \text{For every induced connected subgraph } H \text{ of } G, \\ H^* \text{ has at most two extreme points} \end{cases} \qquad \blacksquare$$

A *tree* is a connected graph containing no cycle. A *spanning tree* of a graph G is a (partial) subgraph of G that is a tree and having the same vertex set as G. It is well known that every connected graph has a spanning tree. A *spanning forest* F of a graph G is a spanning subgraph such that, for every component G' of G, F restricted to G' is a spanning tree of G'.

A *tree-clique* (or *arba*) *graph* is a graph G having a spanning forest F such that every clique of G (thought of as a set of vertices) induces a connected subgraph in F. We call F a *compatible forest* for G. The components of F are called *compatible trees* for the corresponding components of G.

The following theorem, which follows from Theorem 1 of Roberts (1971) and which was noted by Gutierrez and Oubiña (1996), characterizes indifference graphs in terms of tree-clique graphs. We obtain the analogous result for higher dimension indifference graphs in section 4.

Theorem 1.3: Suppose G is a graph. Then

$$G \text{ is an indifference graph} \Longleftrightarrow \begin{cases} G \text{ is a tree-clique graph having} \\ \text{a path as a compatible tree} \end{cases}$$

(In this case the path is *hamiltonian* – it contains every vertex.) $\blacksquare$

We use the abbreviation 'WLOG' to mean 'without loss of generality'. The abbreviation 'iff' means 'if and only if'.

2 Graph Structure and operations

A *gridline indifference graph* (or *GIG* for brevity) is a graph that can be realized in the plane with vertices adjacent whenever they are on a common vertical or horizontal line and are at most some fixed distance ϵ apart, $\epsilon > 0$. A *line* always refers to a vertical or horizontal line, or, more generally, a line that is parallel to one of the axes.

For most of this paper we restrict ourselves to GIG's that are triangulated. In that case, a GIG cannot "grow back into itself". This restriction is consistent with indifference graphs, which are triangulated and the applications of which often deal with the dimension of time.

Two facts about GIG's are immediate: First, any GIG can be realized in the plane with vertices only at positive integral points and using an appropriate ϵ – hence the name. Second, we can use a fixed distance ϵ_x for the horizontal direction and a fixed distance ϵ_y for the vertical direction where $\epsilon_x \neq \epsilon_y$.

A *p-dimensional gridline indifference graph* (or p-d *GIG*, for brevity), where $p \in \mathbb{N}$, is a graph that can be realized in $\mathbb{R}^p$ with vertices adjacent whenever they are on a common line that is parallel to some axis and the distance between them is at most ϵ, where $\epsilon \in \mathbb{R}^+$ is a fixed number.

A *line* always refers to a line parallel to some axis. Like GIG's, any p-d GIG can be realized with vertices only at positive integral points, and we can use a different ϵ_i for each dimension i, $i = 1, 2, \ldots, p$.

Every use of the term *realization* refers to a realization of a (p-d) GIG, unless specified otherwise. Given a realization, any use of ϵ refers to the ϵ specified in the definition above.

Vertex c is *between* a and c in a realization whenever a, b, and c are pairwise adjacent and b is (geometrically) between a and c. Two vertices are *beside* each other in a realization whenever they are adjacent and there is no vertex between them.

For some graphs, betweenness can be viewed in an abstract way. Vertex b is *betwixt* a and c whenever a, b, and c are pairwise adjacent and $|a\nabla c| > \max\{|a\nabla b|, |b\nabla c|\}$, where $x\nabla y := \{v \in V : v \text{ is adjacent to } x \text{ or } y \text{ but not both}\}$. Goodman (1977) shows that in a reduced indifference graph, betweenness and betwixtness are equivalent concepts. Roberts (1971) uses the concepts of besideness and betwixtness in reduced graphs to characterize indifference graphs.

In the remainder of this section we give some important definitions and observations about realizations, many of which are intuitive but nevertheless require formal statements. All observations in this section are stated without proof; proofs can be found in Peterson (1995).

Observation 2.1 (Weak Mapping): Suppose G is a realization. Then the following two statements hold.

(1) Vertex b is between a and $c \Longrightarrow \{a, b, c\}$ induces a *triangle* (a complete subgraph on three vertices)

(2) $\left.\begin{array}{l}\bullet\ \text{Vertex } b \text{ is geometrically between} \\ \quad \text{vertices } a \text{ and } c \\ \bullet\ a \text{ and } b \text{ are nonadjacent}\end{array}\right\} \Longrightarrow a \text{ and } c \text{ are nonadjacent}$ ∎

The term 'weak mapping' comes from the *weak mapping rule* (Goodman, 1977), and its use here is justified by Roberts (1971). Suppose W is a linear order on the vertices of a graph G. After Roberts (1971), we say that W is *compatible* with G whenever $(aWbWc \wedge ac \in E(G)) \Rightarrow ab, bc \in E(G)$. Roberts (1971) shows that a graph G is an indifference graph iff G is compatible with some linear order. Observation 2.1 is exactly the compatibility condition, and Roberts (1971) observes that this is equivalent to Goodman's weak mapping rule.

The l *component of vertex* v, denoted $G_l^{(v)}$, where G is a realization and l is a line containing v in G, is the subgraph of G obtained as follows: Remove all edges incident to v on lines other than l, and take the component containing v.

If l is the horizontal [resp., vertical] line containing v then we denote $G_l^{(v)}$ as $G_{\mathrm{h}}^{(v)}$ [resp., $G_{\mathrm{v}}^{(v)}$].

Observation 2.2 (Triangulation): Suppose G is a triangulated realization with vertex a. Then

$$\left.\begin{array}{l} l_1 \text{ and } l_2 \text{ are different} \\ \text{lines containing } a \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} G_{l_1}^{(a)} \setminus \{a\} \text{ and } G_{l_2}^{(a)} \setminus \{a\} \text{ are disjoint and} \\ \text{nonadjacent, i.e. there is no edge connecting them} \end{array}\right.$$

(Informally, this says that a triangulated p-d GIG cannot "grow back into itself".) ∎

Observation 2.3 (Nonrigidity): Suppose G a is a triangulated realization. Then we may assume no two adjacent vertices have distance exactly ϵ apart. ∎

We will assume nonrigidity for any triangulated realization.

We now proceed to definitions and three observations about operations on realizations: *slide, bend*, and *concatenation.* A *direction about vertex v*, where v is in a realization, is a ray that is parallel to some axis and has endpoint v. For example, the two directions about v on its horizontal line are the left and the right.

A direction about a vertex will sometimes be thought of as the x_i or $-x_i$ direction for some component x_i. Thus, if d is a direction about v and v is translated, we continue to let d be the new direction about v that is parallel to and in the same direction as the former d.

To say that we *shift the realization G about vertex v along direction d*, where d is a direction about v in G, means the following. Let W be the vertices of $\cup\{V(G_l^{(v)}) : l$ is a line containing v but not the line containing $d\}$. Translate each vertex of W the same distance and in the same direction as d.

To say that we *slide the realization G about vertex u to the other side of w*, where u and w are adjacent in G, means to shift G about u along the direction toward w so that u is on the opposite side of w.

To say we *preserve the structure of a realization G* means that, if adjustments are made in that some vertices are transformed to new positions, then the edge set and betweenness relationships remain unchanged and adjacent vertices, modulo at most a translation and/or rotation, remain on the same common line.

Observation 2.4 (Sliding): Suppose G is a triangulated realization with adjacent vertices u and w on line l. Suppose $N_l[u] = N_l[w]$, where N_l is the closed neighborhood restricted to line l. Then G can be slid about u to the other side of w, followed by a finite series of shifts, such that the structure is preserved except betweenness is altered for those triples on l involving both u and some vertex between the original and new positions of u. ∎

The *planar rotation of direction d' onto direction d* in $\mathbb{R}^p$, where $p \geq 2$ and each direction is the plus or minus direction of some component, is the transformation defined as follows: Suppose d' is $\pm x_i$ and d is $\pm x_j$ (their signs may differ). Suppose $\mathbf{x} = (x_1, \ldots, x_p)$. Then $\mathbf{x} \leftarrow \mathbf{x}\mathbf{A}$ where $\mathbf{A}$ is the *planar rotation matrix of d' onto d*, which is the identity matrix except the ij submatrix of $\mathbf{A}$ is

$$\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$$

where the upper right entry $\mathbf{a}_{ij}$ is $+1$ if d' and d agree in sign and -1 otherwise.

To say that we *bend the realization G about vertex u from direction d' onto direction d*, where d' and d are (distinct) directions about u in G, means the following (see Figure 2.1). Define $G_{d'}^{(u)}$ as follows (this is similar to the definition of $G_l^{(u)}$): Remove all edges incident to u on directions about u other than d', and take the component containing u. Then for each $v \in G_{d'}^{(u)}$, transform v by the planar rotation that rotates d' onto d relative to u. That is, $\mathbf{x}_v \leftarrow (\mathbf{x}_v - \mathbf{x}_u)\mathbf{A} + \mathbf{x}_u$ where u is at $\mathbf{x}_u$, v is at $\mathbf{x}_v$, and $\mathbf{A}$ is the planar rotation matrix of d' onto d.

Observation 2.5 (Bending): Suppose G is a triangulated realization, u, d', and d are as in the definition for bending G about u from d' onto d. Suppose also there is a vertex w beside u along d' where w is adjacent only to u along the direction toward u. Let $\bar{d}$ be the direction about u away from w.

Then G can be bent about u from d' onto d, followed by a finite series of shifts, such that the structure is preserved. Moreover, u has no vertex beside it along d', and the vertices connected to u using only edges on $\bar{d}$ in the original realization were not rotated. ∎

The *concatenation* of graphs G and $\hat{G}$, where $V(G) \cap V(\hat{G}) = \emptyset$ and v and $\hat{v}$ are designated vertices of G and $\hat{G}$, respectively, is the graph obtained by identifying v and $\hat{v}$.

Observation 2.6 (Concatenation): Suppose G is a triangulated realization with vertex v where $\mathcal{D} = \{d_1, \ldots, d_q\}$ is the set of directions about v, each having a vertex beside v. Suppose $\hat{G}$ is a triangulated realization with vertex $\hat{v}$ where $\hat{\mathcal{D}} = \{\hat{d}_1, \ldots, \hat{d}_r\}$ is the set of directions about $\hat{v}$, each having a vertex beside $\hat{v}$. Suppose further that $V(G) \cap V(\hat{G}) = \emptyset$ and $\mathcal{D} \cap \hat{\mathcal{D}} = \emptyset$ (here each direction is thought of as the $+x_i$ or $-x_i$ direction for some component i).

Then translating $\hat{G}$ so that v and $\hat{v}$ coalesce and are identified, followed by a certain finite series of shifts, yields a realization of the concatenation of G and $\hat{G}$ that preserves the structures of G and $\hat{G}$ (and adds no new edges). ∎

Every use of the terms *slide, bend* and *concatenate* will be according to the conditions in Observations 2.4 through 2.6. That is, we perform the operation so that the structure is preserved as stated in the corresponding observation.

3 Characterization by Forbidden Subgraphs

This section contains five lemmas followed by theorems that characterize 2-d and p-d GIG's in terms of forbidden subgraphs.

We denote by $\mathcal{E}(\mathcal{C})$, where $\mathcal{C}$ is a set of cliques in a graph G, the *intersection graph* of the edges of the cliques of $\mathcal{C}$. That is, $V(\mathcal{E}(\mathcal{C})) = \mathcal{C}$ and $C_1C_2 \in E(\mathcal{E}(\mathcal{C}))$ whenever C_1 and C_2 are edge-intersecting cliques of G (here C_1 and C_2 are viewed as complete subgraphs of G).

The following lemma will be a crucial element in proving the main results of this chapter.

Lemma 3.1 (Roberts, 1969):

$$G \text{ is a triangulated graph} \Longrightarrow \left\{ \begin{array}{l} G \text{ contains an extreme point (in fact, two non-} \\ \text{adjacent extreme points if } G \text{ is not complete)} \end{array} \right. \qquad \blacksquare$$

The simple proofs of the following two lemmas are omitted; proofs can be found in Peterson (1995).

Lemma 3.2: Suppose $\mathcal{C}$ is a set of cliques in a p-d GIG G. Then

$$\mathcal{E}(\mathcal{C}) \text{ is connected} \Longrightarrow \text{The vertices in } \mathcal{C} \text{ induce an indifference graph}$$

(In this case, $\mathcal{E}(\mathcal{C})$ is a path.) In particular, a complete subgraph in a p-d GIG is realized on a line. ∎

We will use the last statement of the lemma often and without explicit mention.

Lemma 3.3: Suppose G is a triangulated realization with vertex v, and l is a line containing v. Then the following two statements hold.

(1) Any adjacency between $G_l^{(v)}$ and $G \setminus G_l^{(v)}$ is by an edge incident to v (by an edge that is not along l).

(2) Any path from v whose first edge is along line l is in $G_l^{(v)}$. ∎

Figure 3.1 shows four graphs: A *4-fan* (also called a *gem*), *pyramid* (also called $K_{1,1,3}$), *cat*, and *devil*. (The darkened edges are meaningless in this section.) We denote by $\mathbf{F}$ a family of graphs consisting of these four graphs and a G_0.

Lemma 3.4: Suppose G is a graph and $p \in \mathbb{N}$. Then

$$G \in \mathbf{F} \Longrightarrow G \text{ is not a } p\text{-d GIG}$$

Proof: Suppose G is any of the five graphs in $\mathbf{F}$ and $\mathcal{C}$ is the set of cliques in G. Observe that $\mathcal{E}(\mathcal{C})$ is connected, so by Lemma 3.2 if G is a p-d GIG then it is an indifference graph. But G is either a G_0, or it contains an induced claw (if G is a 4-fan, pyramid, or cat) or an induced net (if G is a devil), violating Theorem 1.1. ∎

We denote by $\mathbf{D_2}$ a class of graphs, each having a designated vertex called a *center point*, defined inductively as follows:

Define $\mathbf{D_2}^{(0)}$ as the set of two graphs I and II shown in Figure 3.2. For each the center point is the vertex labeled c.

Define $\mathbf{D_2}^{(i)}$ for $i \in \mathbb{N}$ as $\mathbf{D_2}^{(i-1)} \cup \mathbf{L}^{(i)}$ where $\mathbf{L}^{(i)}$ is the set of graphs having the induced form III or IV shown in Figure 3.3. For each the center point is the vertex labeled c. The graphs $D^{(i-1)}$, $D_1^{(i-1)}$, and $D_2^{(i-1)}$ shown are each a graph in $\mathbf{D_2}^{(i-1)}$ with center points u, u, and w, respectively. (The form of the graph shown is induced, so, for example in Figure 3.3, no vertex of III explicitly depicted, except for u, is contained in or adjacent to $D^{(i-1)}$.)

Define $\mathbf{D_2} := \mathbf{D_2}^{\infty} = \cup\{\mathbf{D_2}^{(i)} : i \in \mathbb{N}\}$.

Lemma 3.5: Suppose G is a triangulated GIG with vertex c. Then

$$\left.\begin{array}{l}\text{In every (2-d) realization } c \text{ has at least two}\\ \text{vertices beside it, that is, } c \text{ has neighbors}\\ \text{on at least two directions about } c\end{array}\right\} \iff \left\{\begin{array}{l} G \text{ contains an induced}\\ D \in \mathbf{D_2} \text{ with center point } c\end{array}\right.$$

Moreover, in this case, there are two vertices beside c in any realization of D.

Proof: Every use of the term *realization* in this proof refers to a 2-d GIG realization. Recall that we are assuming nonrigidity (Observation 2.3).

($\Rightarrow$) We prove this by induction on the number of vertices in G. If G has only three vertices, one of which is c and the other two of which are beside c in any realization, then it is immediate that G must be a I.

Suppose G has more than three vertices. Take a realization of G; by hypothesis c has two vertices beside it. Suppose G is I-free where c is the center point – that is, c is simplicial. Then all vertices adjacent to c are themselves adjacent, and are on the same line in the given realization, WLOG a horizontal line. Take (the unique) two such vertices separated by maximum distance. Call these vertices u and w, and say u and w are to the left and to the right of c, respectively. See Figure 3.4.

Suppose u has no vertex beside it to its left. Then, using weak mapping, $N_{\mathrm{h}}[u] = N_{\mathrm{h}}[c]$ where N_{h} is the closed neighborhood restricted to the horizontal line containing $\{u, c, w\}$, so by Observation 2.4 we can slide G about c to the left of u. Then c has only one vertex beside it, a contradiction. Thus u has a vertex u' beside it to its left and, similarly, w has a vertex w' beside it to its right.

If u' and w' are each adjacent to some neighbor of c other than u and w, respectively, then we have a II. (By weak mapping, the neighbors of u' and the neighbors of w' are distinct.)

Suppose exactly one of u' and w', WLOG w', is adjacent to some neighbor of c other than u or w. Then the only neighbor of u' to the right is u. See Figure 3.3, graph form III. Discard $G_{\mathrm{v}}^{(u)} \setminus \{u\}$ from the realization, leaving a realization of $G \setminus (G_{\mathrm{v}}^{(u)} \setminus \{u\})$ in which u has no vertical neighbors. Now suppose there is a realization of $G_{\mathrm{v}}^{(u)}$ such that u has at most one neighbor beside it, WLOG below u. By Lemma 3.3 (1), the only edges between $G_{\mathrm{v}}^{(u)}$ and $G \setminus (G_{\mathrm{v}}^{(u)} \setminus \{u\})$ are incident to u. Thus, by Observation 2.6, we can concatenate the realizations of $G_{\mathrm{v}}^{(u)}$ and $G \setminus (G_{\mathrm{v}}^{(u)} \setminus \{u\})$ and obtain a realization of G in which adjacent vertices from either of the two realizations are, modulo translation, in the same positions in the new realization of G. In particular, u has no neighbor above it. Now by Observation 2.5 we bend G about u from the left onto the direction above, by Observation 2.4 we slide G about c to the left of u, and c has only one vertex beside it, a contradiction. Thus in every realization of $G_{\mathrm{v}}^{(u)}$, u has at least two vertices beside it. By the inductive hypothesis, $G_{\mathrm{v}}^{(u)}$ contains an induced $D \in \mathbf{D_2}$ with center point u. By triangulation, the D is disjoint from and nonadjacent to U, where U is the set of vertices connected to and on the same horizontal line as u. Thus we obtain a graph having form III.

Finally, suppose neither u' nor w' is adjacent to some neighbor of c other than u or w. Then by a similar argument applied to both directions about c we obtain a graph having form IV.

($\Leftarrow$) The proof proceeds by induction on i in $\mathbf{D_2}^{(i)}$. The inductive hypothesis is that in any realization of D, where $D \in \mathbf{D_2}^{(i')}$ for any $i' < i$ and has center point c, there are two vertices beside c in any realization of D. (This will also prove the last statement of the lemma.)

In any realization of I, it is immediate that c must have two vertices beside it. In any realization of II all vertices are on a single line, by Lemma 3.2. The pair $\{w, w'\}$ must be on the same side of c because w' is adjacent to w and not to c. Similarly $\{u, u'\}$ must be on the same side of c. By weak mapping the two pairs cannot be on the same side. This establishes the basis step.

Now suppose $i \geq 1$. Consider any realization of III. The vertices right of c in Figure 3.3 must all be on the same side of c in any realization because they are adjacent to w', which is not adjacent to c. WLOG they are on the right side, as shown. Thus c has a vertex beside it to the right. The vertex u' must, by weak mapping, be adjacent to u on some direction about u other than the direction toward c. By the inductive hypothesis the realization of $D^{(i-1)}$ has two vertices beside u that, by weak mapping, are in directions about u other than the right and the direction toward u'. Thus u cannot be right of c or five directions about u would be required, which we don't have. Thus c has a vertex beside it to the left.

The argument for a graph having form IV is similar. ∎

Define $\mathbf{E_2}$ as the class of graphs having the induced form shown in Figure 3.5, where D_1

and D_2 are each a graph in $\mathbf{D_2}$ and they intersect only at their common center point c. The vertex labeled c is the common center point for D_1 and D_2.

We can now give a structure theorem for triangulated GIG's. The *degree* of a vertex is the number of edges incident to it. A *pendant vertex* is a vertex with degree one.

Theorem 3.6: Suppose G is a triangulated graph. Then

$$G \text{ is a GIG} \iff G \text{ is } \mathbf{F}\text{- and } \mathbf{E_2}\text{-free}$$

Proof: Every use of the term *realization* in the proof refers to a realization of a (2-d) GIG.

($\Rightarrow$) By Lemma 3.4, G is $\mathbf{F}$-free, since any induced subgraph of a GIG is a GIG. Now suppose $E \in \mathbf{E_2}$, as shown in Figure 3.5. By Lemma 3.5, the D_1 and D_2 each require two directions about the center point c. Since D_1 and D_2 are disjoint and nonadjacent, by weak mapping none of these directions are the same. Thus the D_1 and D_2 require four directions about c. Again by weak mapping, the other neighbor of c must be along another direction about c, which we don't have.

($\Leftarrow$) Suppose G is $\mathbf{F}$- and $\mathbf{E_2}$-free. We use induction on the number of vertices. By Lemma 3.1, G contains an extreme point a. Let C be the clique containing a, and let $C' = C \setminus \{a\}$. Let Z be the set of vertices adjacent to at least two vertices of C but not adjacent to a – these are the vertices that can be z in the definition of extreme point above. Now by the inductive hypothesis $G' := G \setminus \{a\}$ is realizable as a GIG. We may assume there is no other vertex a' in G with the same closed neighborhood as a. For otherwise a' is also simplicial and thus all its neighbors must be realized on a single line; by nonrigidity

(Observation 2.3) we can place a close to a' in a realization of G' and obtain a realization of G. Thus, for any two vertices $x, y \in C'$, there is a $z \in Z$ such that $xz, yz \in G$ and $az \notin G$. There are five cases.

Case 1: $|C'| = 1$, that is, a has only one neighbor x. If there is a realization of G' such that x does not have four neighbors beside it, then we can place a in the vacant direction about x such that it is adjacent only to x. Suppose in every realization of G', x has four neighbors beside it. Take a realization of G'. Then $G_{\mathrm{v}}'^{(x)}$ contains the two vertical neighbors of x. Discard $G_{\mathrm{v}}'^{(x)} \setminus \{x\}$ from the realization, leaving a realization of $G' \setminus (G_{\mathrm{v}}'^{(x)})$ in which x has no vertical neighbor. Now suppose $G_{\mathrm{v}}'^{(x)}$ contains no induced $D_1 \in \mathbf{D_2}$ with x as center point. Then, by Lemma 3.5, there is a realization of $G_{\mathrm{v}}'^{(x)}$ such that x has at most one neighbor beside it, WLOG below it. By Lemma 3.3 (1), the only edges in G' between $G_{\mathrm{v}}'^{(x)}$ and $G' \setminus (G_{\mathrm{v}}'^{(x)} \setminus \{x\})$ are incident to x. But the concatenation of the realizations of these two graphs at x, according to Observation 2.6, yields a realization of G' in which x has no neighbor beside and above it. This violates our assumption that x has four vertices beside it in every realization. Thus $G_{\mathrm{v}}'^{(x)}$ contains an induced $D_1 \in \mathbf{D_2}$ with x as center point. Similarly, $G_{\mathrm{h}}'^{(x)}$ contains an induced $D_2 \in \mathbf{D_2}$ with x as center point. By triangulation, the D_1 and D_2 are disjoint and nonadjacent except by their common center point x. These with a yield an induced $E \in \mathbf{E_2}$, a contradiction.

In cases 2 through 5, $|C'| > 1$, and we fix a realization of G'. WLOG C' is on a horizontal line. Let x and y be the leftmost and rightmost vertices of C', respectively. Observe that a is adjacent at most to vertices between x and y.

Case 2: The vertices of C' are all together – that is, no vertex not in C' is between any vertices of C' – and every vertex of Z is on the same side of C'. WLOG Z is on the right side of C'. Figure 3.6 depicts case 2. Let z be the vertex beside y to the right. Since $xz' \in G$ for some $z' \in Z$ and z' is not left of z, we have by weak mapping that $xz \in G$. Now if x has no vertex beside it to the left then by nonrigidity (Observation 2.3) we can place a just left of x, and we have a realization of G. Thus we may assume x has a vertex u beside it to the left. Since u is left of C' we have that $u \notin Z$, so u is adjacent only to x to the right. Suppose $G_{\mathrm{v}}'^{(x)}$ contains no induced $D \in \mathbf{D_2}$ with x as center point. We proceed as in case 1 to obtain a realization of G' where x has no vertex beside and above it, and the betweenness relationships on the horizontal line containing x are preserved. Now by Observation 2.5 we bend G' about x from the left onto the direction above, then place a left of x, and we have a realization of G. Thus we may assume $G_{\mathrm{v}}'^{(x)}$ contains an induced $D \in \mathbf{D_2}$ with x as center point. By triangulation, the D is disjoint from and nonadjacent to $\{u, z\}$ and C', except by its center point x. But then $\{u, x, D, a, z\}$ induces an $E \in \mathbf{E_2}$ where $D_1 = D$ and D_2 is a I, a contradiction.

Case 3: The vertices of C' are all together in the realization, there are vertices of Z on both sides of C', and there are (at least) two nonadjacent vertices of Z. By weak mapping, any two nonadjacent vertices of Z are on opposite sides of C'. Figure 3.7 depicts case 3. Take a maximal subset of vertices Z' of Z that are all adjacent to x, y, and each other. Observe $\emptyset \neq Z' \subsetneq Z$ since some vertex of Z is adjacent to x and y and by case assumption. Then, using weak mapping, there is WLOG a vertex of Z' to the right of y and the leftmost vertex z_1 of Z is not in Z'. Let z_2 be the rightmost vertex of Z'. By maximality of Z' and weak mapping, $z_1 z_2 \notin G$. By definition of Z, $z_1 x, z_1 y' \in G$ for some vertex $y' \in C' \setminus \{x\}$ (possibly $y' = y$). Now $y' z_2 \in G$ since $x z_2 \in G$ and by weak mapping. But then $\{z_1, x, a, y', z_2\}$ induces a pyramid, a contradiction. Thus case 3 cannot occur.

Case 4: The vertices of C' are all together in the realization, there are vertices of Z on *both sides of C', and all vertices of Z are pairwise adjacent.* Figure 3.8 depicts case 4. Let z_1 be the leftmost vertex of Z and z_2 be the rightmost vertex of Z. By case assumption $z_1 z_2 \in G$, and by weak mapping all vertices in between are adjacent.

Case 4a: Suppose z_1 has no neighbor to the left and z_2 has no neighbor to the right. By Observation 2.4 we can slide G' about each of the vertices of Z to one (either) side of C' and we are in case 2. Thus we may assume that case 4a does not hold, so for the rest of case 4 we assume WLOG that z_1 has a neighbor u beside it to the left.

Case 4b: Suppose $ux \in G$. Since $u \notin Z, uy \notin G$. But then, using weak mapping, we have that $\{u, z_1, x, a, y, z_2\}$ induces a cat, a contradiction. Thus case 4b cannot hold, so for the rest of case 4 we have that $ux \notin G$.

Now we may assume that z_2 has a vertex w beside it to the right, for if not then by weak mapping and Observation 2.4 we can slide G' about each of the vertices of Z that are right of y to the left of x, and we are in case 2. By an argument symmetrical to case 4b just above, we may assume that $yw \notin G$.

Case 4c: Suppose $uz_1' \in G$ for some $z_1' \in Z \setminus \{z_1\}$ and $z_2'w \in G$ for some $z_2' \in Z \setminus \{z_2\}$. But then, using weak mapping, we have that $\{u, z_1, z_1', x, a, y, z_2', z_2, w\}$ induces a devil, a contradiction.

Case 4d: Suppose $uz_1' \in G$ for some $z_1' \in Z \setminus \{z_1\}$ and w is adjacent only to z_2 to the left. If by Observation 2.5 we can bend G' about z_2 from the right onto some other direction, then by Observation 2.4 we can slide G' about each of the vertices that are right of y to the left of x, and we are in case 2. Otherwise, by the same argument we used in cases 1 and 2, there is an induced $D \in \mathbf{D_2}$ that is disjoint from and nonadjacent to $\{u, z_1, z_1', w\}$ and C', except by its center point z_2. But then $\{u, z_1, z_1', x, a, y, z_2, w, D\}$ induces an $E \in \mathbf{E_2}$ where

w is the pendant vertex, $D_1 = D$, and D_2 is a II (u and a are the two vertices of degree two in II), a contradiction.

By a symmetrical argument, the case where $z_2'w \in G$ for some $z_2' \in Z \setminus \{z_2\}$ and u is adjacent only to z_1 to the right leads to a contradiction.

Case 4e: Suppose u is adjacent only to z_1 to the right and w is adjacent only to z_2 to the left. By an argument similar to that in case 4d, we can either bend G' about z_1 from the left onto some other direction or bend G' about z_2 from the right onto some other direction, and get into case 2, or there are induced subgraphs $D', D \in \mathbf{D_2}$ where D' is disjoint from and nonadjacent to $\{u, z_2, w\}$ and C' except by its center point z_1 and D is disjoint from and nonadjacent to $\{u, z_1, w\}$ and C' except by its center point z_2. By triangulation (applied to the vertical and horizontal components of either z_1 or z_2) the D' and D are disjoint and nonadjacent. But then $\{u, D', z_1, x, a, y, z_2, D, w\}$ induces an $E \in \mathbf{E_2}$ where w is the pendant vertex, $D_1 = D$, and D_2 has form III, a contradiction.

Case 5: The vertices of C' are not all together in the realization. Let Z' be the set of vertices between x and y that are not adjacent to a. Figure 3.9 depicts case 5. We may assume that in the given realization Z' is minimum, that is, there is no realization having fewer vertices of Z between the vertices in C'. Thus, G' cannot be slid about any vertex in Z' to the left of x or right of y (unless G' must be simultaneously slid about some other vertex that is not between x and y to a position between x and y).

Let U be the set of vertices adjacent to and to the left of x.

Case 5a: Suppose that $u'y' \in G$ for some $u' \in U, y' \in C' \setminus \{x\}$, and $u'z' \notin G$ for some $z' \in Z'$. But then using weak mapping $\{u', x, y', a, z'\}$ induces a pyramid, a contradiction.

Case 5b: Suppose there is a $u' \in U$ such that $u'z' \in G$ for some $z' \in Z'$ and u' is not adjacent to every vertex in C'; in particular, by weak mapping, $u'y \notin G$. By Observation 2.4 we can slide G' about z' to the left of x, contradicting minimality, unless one of the following holds. (1) There is a vertex $u'' \in U$ such that $u''z' \notin G$. But then $\{u'', u', x, a, z', y\}$ induces a 4-fan, a contradiction. (2) There is a vertex w' (to the right of y) such that $z'w' \in G$ and $xw' \notin G$. But then, using weak mapping, $\{u', x, a, z', y, w'\}$ induces a G_0, a contradiction.

Observe that, since cases 5a and 5b do not hold, if $u' \in U$ is adjacent to any vertex right of x then it is adjacent to y.

Case 5c: Suppose there is a $u' \in U$ such that $u'y \in G$ and there is a $u'' \in U$ such that $u''y \notin G$ (and thus u'' is adjacent to no vertex right of x). But then, using weak mapping, $\{u'', u', x, a, z', y\}$ induces a cat, where $z' \in Z'$.

Case 5d: Suppose that for all $u' \in U, u'$ is adjacent to no other vertex in C' but x. Then $u'w \notin G$ for all $u' \in U, w \in C' \setminus \{x\}$; in particular this is true for $u' = u$ where u is beside x to the left. Suppose that by Observation 2.5 we can bend G' about x from the left onto some direction. We may assume there is no vertex w' (to the right of y) such that $z'w' \in G$ and $xw' \notin G$. For this is symmetric to case 5b, replacing u' and y with w' and x, respectively. Now, by Observation 2.4, we can slide G' about some vertex of Z', say the leftmost vertex z', to the left of x. But this would contradict minimality. Thus we cannot bend G' about x from the left onto any direction. Again by an argument similar to that in case 4d (and cases 1 and 2), there is an induced subgraph $D \in \mathbf{D_2}$ that is disjoint from and nonadjacent to $\{u, z'\}$ (where $z' \in Z'$) and C' except by its center point x. But then $\{u, x, D, a, z'\}$ induces an $E \in \mathbf{E_2}$ where $D_1 = D$ and D_2 has form I, a contradiction.

Case 5e: Suppose that $u'y \in G$ for every $u' \in U$; by symmetry on what we have already proved we may assume $xw' \in G$ for every $w' \in W$ where $W :=$ the set of vertices adjacent to and to the right of y. Then, by Observation 2.4, we can slide G' about any $z' \in Z'$ to the left of x (or to the right of y), violating minimality.

This completes case 5 and the proof. ∎

We extend our work above to higher dimensions.

Suppose $p \in \mathbb{N}$ is given. Define $\mathbf{D_p}$ the same as in the definition of $\mathbf{D_2}$, except $\mathbf{L}^{(i)}$ is the set of graphs having the induced form III' or IV' shown in Figure 3.10. For each the center point is the vertex labeled c. The graphs $D_j^{(i-1)}, j = 1, \ldots, p-1$ in III' are each a graph in ${\mathbf{D_p}}^{(i-1)}$ and they pairwise intersect only at their common center point u. The graphs $D_{1,j}^{(i-1)}$ [resp., $D_{2,j}^{(i-1)}$], $j = 1, \ldots, p-1$ in IV' are each a graph in ${\mathbf{D_p}}^{(i-1)}$ and they pairwise intersect only at their common center point u [resp., w]. If $p = 1$ then the D_j's (in III') and the $D_{1,j}$'s and $D_{2,j}$'s (in IV') are vacuous.

Define $\mathbf{D_p} := {\mathbf{D_p}}^{\infty} = \cup\{{\mathbf{D_p}}^{(i)} : i \in \mathbb{N}\}$.

Define $\mathbf{E_p}$ as the class of graphs having the induced form shown in Figure 3.11, where each $D_j, j = 1, \ldots, p$ is a graph in $\mathbf{D_p}$ and they pairwise intersect only at their common center point c. The vertex labeled c is the common center point for the D_j's.

The following theorem characterizes triangulated p-d GIG's.

Theorem 3.7: Suppose G is a triangulated graph and $p \in \mathbb{N}$. Then

$$G \text{ is a } p\text{-d GIG} \iff G \text{ is } \mathbf{F}\text{- and } \mathbf{E_p}\text{-free}$$

Proof: Suppose $p = 1$. ($\Rightarrow$) Theorem 1.1 implies that G is $\mathbf{F}$-free, since a 4-fan, a pyramid, and a cat each contain an induced claw and a devil contains an induced net. Now observe that $\mathbf{D_1}$ contains, up to isomorphism, only four graphs: Graphs I, II, III' (with the D_j's vacuous), and IV' (with the $D_{1,j}$'s and $D_{2,j}$'s vacuous). It follows that any $E \in \mathbf{E_1}$ contains an induced claw or net, and Theorem 1.1 implies that G is $\mathbf{E_1}$-free. ($\Leftarrow$) This also follows from Theorem 1.1, since $\mathbf{F}$ contains a G_0, a graph $E \in \mathbf{E_1}$ with D_1 having form I is a claw, and a graph $E \in \mathbf{E_1}$ with D_1 having form IV' is a net.

Suppose $p > 1$. We first adjust the proof of Lemma 3.5 so that it holds for G a triangulated p-d GIG with vertex c, as follows. In the ($\Rightarrow$) direction the argument proceeds exactly as in the proof there until we reach the case where only one of u' and w' is adjacent to some neighbor of c other than u or w. Recall that, by the concatenation argument, $G_{\mathrm{v}}^{(u)}$ contains an induced $D \in \mathbf{D_p}$ with center point u that is disjoint and nonadjacent to U, where U is the set of vertices connected to and on the same horizontal line as u. Now we apply this argument not just to $G_{\mathrm{v}}^{(u)}$ but to $G_l^{(u)}$ for each line l containing u, except the horizontal line. We obtain $D_1, \ldots, D_{p-1} \in \mathbf{D_p}$ where for $j = 1, \ldots, p-1$, D_j and U are disjoint and nonadjacent except by u. By triangulation, when $i \neq j$, D_i and D_j are disjoint and nonadjacent except by u. Thus we obtain a graph having form III'. If neither u' nor w' is adjacent to some neighbor of c other than u or w, then by a similar argument we obtain a graph having form IV'.

In the ($\Leftarrow$) direction the argument is the same as given in the proof of Lemma 3.5 with the following modifications. For a graph G having form III', the $D_j^{(i-1)}$'s require $2p-2$ directions, so sliding G about u to the right of c would require $2p+1$ directions about u, which we don't have. For a graph having form IV' the argument is similar. Thus Lemma 3.5 holds for G a triangulated p-d GIG with vertex c.

The proof of Theorem 3.7 proceeds like that of Theorem 3.6, using the extended version of

Lemma 3.5. The only modifications are that, each time we use the concatenation argument (in cases 1, 2, 4d, 4e, and 5d), we apply it $p-1$ times instead of just once, similar to the way we extended the proof of Lemma 3.5. In those cases we obtain a III' (instead of a III) or a IV' (instead of a IV). ∎

We conclude the section with the following theorem.

Theorem 3.8: Suppose G is a triangulated graph. Then

$$G \text{ is a } p\text{-d GIG, some } p \in \mathbb{N} \Longleftrightarrow G \text{ is } \mathbf{F}\text{-free}$$

Proof: ($\Rightarrow$) This follows from Theorem 3.7. ($\Leftarrow$) This also follows from Theorem 3.7 by taking $p \geq |V(G)|$, in which case a graph $E \in \mathbf{E_p}$ cannot occur. ∎

4 Characterization by Tree-Clique Graphs

This section contains four propositions that characterize or partially characterize types of tree-clique graphs. Theorem 4.5 then characterizes GIG's in terms of a type of tree-clique graph.

Recall the definition of tree-clique graph from section 1. Observe that G_0 is not a tree-clique graph: It is a straightforward check that no spanning tree is compatible. A 4-fan, a pyramid, a cat, and a devil are each tree-clique graphs. The darkened edges in Figure 3.1

indicate a compatible tree for each. Theorem 1.1 characterized indifference graphs in terms of a class of tree-clique graphs. Thus tree-clique graphs extend indifference graphs, but in a different way than do p-d GIG's.

We introduce two variations of tree-clique graphs. A *tree-clique path graph* is a tree-clique graph G having a compatible forest F for which every clique of G induces a path in F. We call F a *path compatible forest*, and the components of F we call *path compatible trees* for the corresponding components of G. A *tree-clique indifference graph* is a tree-clique path graph G having a path compatible forest F for which any two edge-intersecting cliques of G induce a path in F. We call F an *indifference compatible forest*, and the components of F we call *indifference compatible trees* for the corresponding components of G.

The graphs in Figure 3.1 are tree-clique path graphs; the darkened edges indicate the path compatible trees. They however are not tree-clique indifference graphs.

A p-d GIG is a tree-clique graph, but the converse is not true. Indeed, unlike p-d GIG's, an induced subgraph of a tree-clique graph need not be a tree-clique graph. For example, take any connected graph G that is not a tree-clique graph (an example is a G_0), add a vertex, and join it to every vertex in G. The result is a tree-clique graph – the added edges constitute a compatible tree. Letting G be a 4-cycle shows that the same is true for tree-clique path graphs. Thus there need not be a realization for a tree-clique graph in the same sense as there is for a p-d GIG, in which every induced subgraph of a realization is also a p-d GIG. If, however, a tree-clique path graph is *triangulated* then every induced subgraph is a tree-clique path graph (Peterson, 1995).

We say that G/F *contains an induced* J, where G is a graph, F is a spanning forest of G, and J is a graph with certain edges designated as forest edges, whenever some induced subgraph of G is (a graph isomorphic to) J such that the designated edges are in F and the other edges are not in F. We say that *G/F is J-free* whenever G/F contains no induced J.

We denote by J_i, $i = 1, 2, 3, 4$, the graphs in Figure 4.1, where darkened edges indicate forest edges.

Proposition 4.1:

$$\left.\begin{array}{ll} \text{(a)} & G \text{ is a tree-clique graph} \\ \text{(b)} & F \text{ is a compatible forest} \end{array}\right\} \Longrightarrow G/F \text{ is } J_1\text{- and } J_2\text{-free}$$

Proof: Suppose D is an induced *diamond* of G. (A diamond is a complete graph on four vertices, minus one edge.) Let x, y be the vertices of degree two and u, v be the other vertices in D. See J_1 and J_2 in Figure 4.1. It suffices to show that the two edges incident to x in D cannot both be in F. Suppose in contradiction that these edges are in F. Now there is some clique A of G containing triangle yuv. Then there is a (u, v)-path in F using only vertices in A, since these vertices induce a connected subgraph of F. Since $x \notin A$, this path together with the two edges adjacent to x in D yield a cycle in F, a contradiction. ∎

The converse of Proposition 4.1 is false, even for triangulated graphs. That is, if a graph is J_1- and J_2-free with respect to a given spanning forest, the spanning forest need not be compatible. See Figure 4.2.

Proposition 4.2: Suppose G is a tree-clique graph. Then

$$\left.\begin{array}{ll} \text{(a)} & G \text{ is a tree-clique path graph} \\ \text{(b)} & F \text{ is a path compatible forest} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} F \text{ is a compatible forest for } G \\ \text{such that } G/F \text{ is } J_3\text{-free} \end{array}\right.$$

Proof: ($\Rightarrow$) Suppose G/F contains an induced J_3. See Figure 4.1; we use the vertex labeling shown there. Some clique A of G contains the four vertices. Then, because of vertex x in J_3, it is impossible for A to induce a path in F.

($\Leftarrow$) Suppose F is a compatible forest but is not path compatible. Then some clique A of G induces a tree T' in F that is not a path. It follows that some vertex $x \in A$ has degree three in $A \cap T'$, which yields a J_3. ∎

Proposition 4.3: Suppose G is a tree-clique path graph. Then

$$\left.\begin{array}{ll}\text{(a)} & G \text{ is a tree-clique indifference graph} \\ \text{(b)} & F \text{ is an indifference compatible forest}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} F \text{ is a path compatible forest} \\ \text{for } G \text{ such that } G/F \text{ is } J_4\text{-free}\end{array}\right.$$

Proof: ($\Rightarrow$) Suppose G/F contains an induced J_4. See Figure 4.1; we use the vertex labeling shown there. There is some clique A of G containing triangle xyu, and some clique B of G containing triangle xyv. But because of vertex x in J_4, it is impossible for $A \cup B$ to induce a path in F.

($\Leftarrow$) Suppose F is a path compatible forest and A and B are two edge-intersecting cliques of G. We first argue that the subgraph P of F induced by the vertices of $A \cap B$ is a path. Suppose not. Observe that $A \cap B$ is a complete subgraph in G. By Proposition 4.2 $(A \cap B)/P$ has no induced J_3, that is, no vertex has degree three in P. Thus P is the union of (at least two) vertex disjoint paths. Take two vertices s and t that are disconnected in P. By definition of compatible forest, there is an (s,t)-path in F using only vertices of A. Since s and t are disconnected in P, this (s,t)-path must include some vertex $a \in A \setminus B$. Similarly, there is an (s,t)-path in F using only vertices of B. But then there are two (s,t)-paths in F that are different (because of a), which implies that a cycle in F exists. This is a contradiction, so P is a path.

Now suppose F is not indifference compatible. Suppose $A \cup B$ does not induce a path in F. See Figure 4.3. Using that F is path compatible and P is a path, we see that for (at least) one of the endvertices x of P, $xu, xv \in T$ for some $u \in A \setminus B$ and $v \in B \setminus A$. Let y be the vertex adjacent to x in P. By Proposition 4.2, $\{x, y, u, v\}$ cannot induce a J_3 in G/F, so $uv \notin G$. But then these vertices induce a J_4. ∎

A *walk* of a graph is a finite sequence of vertices where each consecutive pair of vertices is adjacent. Thus, a path is a walk in which no vertex appears twice. The *endvertices* are the first and last vertices. Every vertex that appears between the endvertices are *interior* vertices. (In a walk, a vertex may be both an endvertex and an interior vertex.) Vertices that are next to each other in the sequence are *consecutive*. It is well known that the endvertices of a walk are the endvertices of some path that uses only (but not necessarily all) vertices of the walk.

We present one more proposition, on triangulated tree-clique path graphs, before proceeding to the main result of the section.

Proposition 4.4: G is a tree-clique indifference graph $\Longrightarrow$ G is triangulated

Proof: Suppose G has a hole Z and T is a path compatible tree for the component containing Z. We show T is not indifference compatible. Let $Z = x_0 \ldots x_{n-1}$. Edge x_0x_1 is in a clique containing no vertices in Z except x_0 and x_1. Thus there is an (x_0, x_1)-path $P_{0,1}$ in T using no other vertices in Z, and the vertices of $P_{0,1}$ are a complete subgraph in G. Similarly there is an (x_i, x_{i+1})-path $P_{i,i+1}$ in T for each i ('each i' means $i = 0, \ldots, n-1$, and all addition is modulo n).

We first show that $P_{0,1}$ and $P_{1,2}$ share an edge incident to x_1. Suppose in contradiction that $x_1v \in P_{0,1}$ and $x_1w \in P_{1,2}$ where $v \neq w$. Using the rest of $P_{1,2}$ from w to x_2, then $P_{i,i+1}, i = 2, \ldots, n-1$, and then $P_{0,1}$ from x_0 to v, we obtain a (w, v)-walk in T not involving x_1. Thus there is a (w, v)-path in T not involving x_1. But this with x_1 yields a cycle in T, a contradiction. (Note that this implies $x_0x_1 \notin T$, otherwise $P_{0,1}$ consists of edge x_0x_1, which is impossible since $P_{1,2}$ does not use x_0.) Similar results hold for each x_i.

Suppose that, for some i and j, the internal vertices for $P_{i,i+1}$ differ from the internal vertices for $P_{j,j+1}$. Then WLOG $P_{0,1}$ and $P_{1,2}$ differ in their internal vertices. See Figure 4.4. Write $P_{0,1}$ as $y_0y_1 \ldots y_py_{p+1}$ (where $y_0 = x_0$ and $y_{p+1} = x_1$) and $P_{1,2}$ as $y_{p+1}y_p \ldots y_{p-l}w \ldots x_2$ where $w \notin \{y_1, \ldots, y_p\}$ and $0 \leq l < p$ (it is possible that $w = x_2$ if $l \neq p-1$). Note that $y_{p-l-1}y_{p-l+1} \in G$ since both vertices are in $P_{0,1}$ and are thus in a common clique. Similarly $wy_{p-l+1} \in G$ since both vertices are in $P_{1,2}$. By Proposition 4.2 using $\{y_{p-l-1}, y_{p-l}, y_{p-l+1}, w\}$, $y_{p-l-1}w \notin G$. Then by Proposition 4.3, T is not indifference compatible, and we are done.

Suppose the internal vertices for the $P_{i,i+1}$'s are all the same. If there is only one (common) internal vertex y for each path then, using that $x_1x_3 \notin G$ since Z is a hole, $\{x_1, x_2, x_3, y\}$ induces a J_4 in G/T. Then by Proposition 4.3, T is not indifference compatible, and we are done. Now suppose there are $p \geq 2$ internal vertices. Then these vertices form a path in T; let the path be $y_1 \ldots y_p$. Observe that, for each i, $x_iy_1 \in T$ or $x_iy_p \in T$. Otherwise $x_iy_q \in T$ for some q, $1 < q < p$, and a cycle in T will necessarily occur. WLOG $x_1y_1 \in T$. Observe that, for each i, $\{x_i, y_1, \ldots, y_p, x_{i+1}\}$ is a complete subgraph in G. If x_0y_1 or $x_2y_1 \in T$, WLOG $x_2y_1 \in T$, then Proposition 4.2 is violated using $\{x_1, x_2, y_1, y_2\}$. Thus $x_0y_p, x_2y_p \in T$. Then, since $x_0x_2 \notin G$ and applying Proposition 4.3 to $\{x_0, x_2, y_p, y_{p-1}\}$, T is not indifference compatible. ∎

Theorem 4.5: Suppose G is a graph. Then the following are equivalent.

(a) G is a triangulated p-d GIG, some $p \in \mathbb{N}$.

(b) G is triangulated and $\mathbf{F}$-free.

(c) G is a tree-clique indifference graph.

Moreover, given any triangulated realization there is an indifference compatible forest F – and vice-versa – such that vertices are beside each other in the realization iff they are adjacent in F.

Proof: The equivalence (a)$\Leftrightarrow$(b) is Theorem 3.8. We prove the last statement of the theorem, which also proves the equivalence (a)$\Leftrightarrow$(c).

To prove the statement without the vice-versa, use the edges between vertices that are beside each other in the given realization of G. Clearly these edges define a spanning subgraph for which there is a path between any two connected vertices. Suppose there is a hole Z. By our construction, there must be a vertex $v \in Z$ such that the edges incident to v in C are on two different lines l_1 and l_2. But by Lemma 3.3 (2), $G_{l_1}^{(v)}$ and $G_{l_2}^{(v)}$ both contain C, violating triangulation. Thus it is a spanning forest. By Lemma 3.2 and weak mapping it is indifference compatible.

To prove the vice-versa part of the statement, we use induction on the number of vertices. Suppose G is not connected. We apply the induction hypothesis to the components of G separately, place the realizations of the components sufficiently far apart so they do not interfere, and we are done. Thus we may assume G is connected and F is a spanning tree T. By Proposition 4.4 G is triangulated, and so by Lemma 3.1 it contains an extreme point a. Let C be the clique containing a. By Proposition 4.2 the degree of a in the given indifference compatible tree T is either one or two. We consider the two cases.

Case 1: The degree of a in T is one. Let x be the neighbor of a in T. Remove a from G to obtain G' and a new spanning tree T'. Let $C' = C \setminus \{a\}$. It is immediate that T' is a compatible tree for G', and by Propositions 4.2 and 4.3 it is indifference compatible. By the induction hypothesis, G' is realizable as a triangulated p-d GIG with vertices beside each other iff they are adjacent in T'. If x is the only vertex of C', that is, a is adjacent only to x in G, then a can be placed beside x using a new dimension and we are done. Thus we may assume $|C'| > 1$. See Figure 4.5. Now x is an endvertex of the path in T' induced by C', else the two vertices of C' adjacent to x in T' together with x and a violate Proposition 4.2. WLOG the other vertices of C' are (directly) to the right of x in the realization of G'. If x has no vertex u beside it to the left, then place a close to and to the left of x (recall that we are assuming nonrigidity, Observation 2.3). Suppose there is such a u. Then $ux \in T'$ and so $ux \in T$ as well. Also $ux' \notin G$ where $x' \in C'$ is the vertex beside x to the right, else Proposition 4.3 is violated using $\{a, u, x, x'\}$. Thus, by Observation 2.5, we can bend G' about x from the left onto some other direction – a new dimension may be required – and we place a close to and to the left of x.

Case 2: The degree of a in T is two. Let x and y be the two neighbors of a in T. By Proposition 4.1, x and y have no common neighbor z where $az \notin G$. Then, by definition of extreme point, at least one of x and y, WLOG x, is simplicial and in the same clique C as a. Now remove a to obtain G' from G and add xy to obtain T' from T ($xy \in G$ because a is simplicial). It is immediate that T' is a spanning tree for G'. To see that it is indifference compatible, suppose two (possibly equal) edge-intersecting cliques of G' induce P' in T'. The corresponding cliques in G induce a path P in T. (Note that corresponding cliques A of G and A' of G' are identical, except where $A = C$ and $A' = C'$.) If P does not include a then it does not include x, so $P' = P$. If P includes a and consequently x, but not y, then a is pendant in P so $P' = P \setminus \{a\}$. If P includes a, x, and y then xay in P is replaced by xy in P'. Thus, in any case, P' is a path so T' is indifference compatible.

By the induction hypothesis, G' is realizable as a triangulated p-d GIG with x and y beside each other. We place a between x and y and close to x to obtain the desired realization. ∎

5 Partial Characterization by Extreme Points

In this section we investigate extreme points and reduced graphs of p-d GIG's, as motivated by Theorem 1.2. We present two propositions. The first characterizes the role of extreme points in p-d GIG's. The second is a partial analogue of Theorem 1.2.

Recall the definition of reduced graph (Section 1) that $[x]$ denotes the set of vertices with the same closed neighborhood as x.

Proposition 5.1: Suppose G is a triangulated p-d GIG, some $p \in \mathbb{N}$, and a is a vertex of G. Then the following are equivalent.

(a) Vertex a is an extreme point of G.

(b) In every realization of G, some $a' \in [a]$ has at most one vertex beside it.

(c) In every indifference compatible forest F for G, some $a' \in [a]$ is a pendant or isolated vertex of F.

Proof: The equivalence (b)$\Leftrightarrow$(c) follows from Theorem 4.5. We show (a)$\Leftrightarrow$(b).

(a)$\Rightarrow$(b) Suppose a is an extreme point. Then all the vertices of $[a]$ are extreme points. WLOG the vertices of $[a]$ are on a horizontal line, and let u and w be the leftmost and rightmost vertices of $[a]$, respectively ($u = w$ if $|[a]| = 1$). Vertex u has no neighbor in any other dimension or else it is not simplicial; similarly for w. Suppose w has vertex y beside it to the right; if not we are done with $a' = w$. Now suppose in contradiction that u has a neighbor x beside it to the left. Since x and y are not in $[a]$, they are each in a clique other than the one containing $[a]$. Since u, w, a all have the same closed neighborhood, $ax, ay \in G$. Then there is a vertex z where $xz, yz \in G$ and $az \notin G$. Then z is on the horizontal line containing $[a]$. But by weak mapping (Observation 2.1) there is no place such a z can exist, a contradiction.

(b)$\Rightarrow$(a) If a is isolated then it is extreme. Thus we may assume a (and each vertex in $[a]$) has at least one vertex beside it. Given a realization, suppose $a' \in [a]$ has only one vertex beside it, WLOG to the right. Then every vertex of $[a]$ is to the right of a'. Let a'' be the rightmost vertex of $[a]$. It suffices to show a' is extreme, for a is extreme iff a' is.

Suppose $a'x, a'y \in G$. Then one of x, y is further to the right, and by weak mapping it follows that $xy \in G$. Thus we have that a' is simplicial.

Now suppose a' is not extreme. Then there are vertices x and y that are each in some clique other than the one containing a', and there is no vertex z such that $xz, yz \in G$ and $a'z \notin G$. We may assume x is right of a''. For suppose x is between a' and a''. Then by weak mapping $N_{\mathrm{h}}[x] = N_{\mathrm{h}}[a'']$, where N_{h} is the closed neighborhood restricted to the horizontal line containing x. Thus, by Observation 2.4, G can be slid about x to the right of a''. Similarly, we may assume y is right of a''. It follows that a'' has a vertex beside it to the right. Also, WLOG y is right of x. Now by the contradictory assumption, any vertex right of y that is adjacent to x is also adjacent to a'. Then $N_{\mathrm{h}}[x] = N_{\mathrm{h}}[a']$, so by Observation 2.4 G can be slid about x to the left of a'. This yields a realization where every vertex in $[a]$ has two vertices beside it. This is a contradiction, so a' is extreme. ∎

For $p > 1$, it is *not* true that *every* vertex with only one vertex beside it in a realization (equivalently, being a pendant vertex in a indifference compatible forest) need be an extreme point. See vertex a in Figure 5.1.

We precede the next proposition with a definition. Recall from Section 3 that, given a set $\mathcal{C}$ of cliques in a graph, $\mathcal{E}(\mathcal{C})$ is the intersection graph of the edges of the cliques of $\mathcal{C}$. A *linked set* of a graph G is a set of cliques $\mathcal{C}$ such that $\mathcal{E}(\mathcal{C})$ is connected.

Proposition 5.2: Suppose G is a graph and $p \in \mathbb{N}$. Then

$$G \text{ is a } p\text{-d GIG} \Longrightarrow \left\{ \begin{array}{l} \text{For every induced subgraph } H \text{ of } G, \text{ any linked set of} \\ H^* \text{ contains at most two extreme points of } H^* \end{array} \right.$$

Proof: It suffices to prove the statement only for G^*, since any induced subgraph of G is also a p-d GIG.

Let $\mathcal{C}$ be a linked set of G^*. We may assume $\mathcal{C}$ is maximal, for if $\mathcal{C}$ contains three extreme points of G^* then so does any maximal linked set containing $\mathcal{C}$. By Lemma 3.2, the vertices in $\mathcal{C}$ induce an indifference graph I^*. We show that for any vertex $a \in I^*$, if a is extreme in G^* then a is extreme in I^*. This will complete the proof, for if more than two extreme points of G^* are in I^* this violates Theorem 1.2.

Suppose a is extreme in G^* and is in clique $C \in \mathcal{C}$. Since a is simplicial in G^* it is simplicial in I^*. Suppose $ax, ay \in I^*$ (so $ax, ay \in G^*$ also) where x, y are each in some clique other than C. Then there is a vertex $z \in G^*$ where $xz, yz \in G^*$ and $az \notin G^*$. Some clique $C' \neq C$ contains triangle xyz, and C' edge-intersects with C at edge xy. By maximality of $\mathcal{C}$ we have that $C' \in \mathcal{C}$ and thus $z \in I^*$. It follows that a is an extreme point of I^*. ∎

For $p > 1$ the converse of Proposition 5.2 is false. A 4-fan and a cat are counterexamples.

6 Conclusion

We have characterized triangulated p-d GIG's in terms of forbidden induced subgraphs. A triangulated graph G is a p-d GIG iff it is 4-fan-, pyramid-, cat-, and devil-free, and it does not contain an induced subgraph that requires $p + 1$ dimensions. We then studied tree-clique, tree-clique path, and tree-clique path-linked graphs. We found partial and full characterizations of these graphs by forbidden subgraphs with respect to their compatible forests. We then connected these concepts with p-d GIG's. A graph is a triangulated p-d GIG for some p iff it is a tree-clique indifference graph. Moreover, there is a natural correspondence between indifference compatible forests and realizations of a p-d GIG. We have also characterized the role of extreme points in p-d GIG's. A vertex is an extreme point in a triangulated p-d GIG iff in every realization some vertex equivalent to it, modulo closed neighborhoods, has at most one vertex beside it. Finally, a partial characterization of p-d GIG's is found in terms of extreme points. If a graph is a p-d GIG then for the reduced graph of every induced subgraph, any linked set contains at most two extreme points.

Some questions remaining are:

(1) *What other conditions are needed to characterize* p-d *GIG's if they are not triangulated or finite?* Peterson (2003) characterizes p-d GIG's where $\epsilon = \infty$ in terms of coloring of the *clique graph* – the intersection graph of the cliques. A similar approach may be applied using the intersection graph of the maximal linked sets.

(2) *Can these results on GIG's be applied to characterize or partially characterize grid graphs?* A *grid graph* is a GIG that can be realized with vertices at integral points and $1 < \epsilon < 2$. Grid graphs arise, for example, in the *frequency assignment problem*, where communication stations are on a grid and there is frequency interference between adjacent grid points. Recognition of grid graphs seems to be a difficult problem (Kennedy and Quintas, 1983) and Clark, Colbourn, and Johnson, 1990).

(3) *How can tree-clique or tree-clique path graphs be characterized by forbidden subgraphs,* in the same way that tree-clique indifference graphs are characterized in Theorem 4.5?

(4) *Is there a full characterization of* p-d *GIG's in terms of extreme points, analogous to Theorem 1.2 for indifference graphs?* Roberts (1969) finds a characterization of indifference graphs in terms of *semiorders* (a type of binary relation, which he uses to prove Theorem 1.2. A similar approach for p-d GIG's may be applied.

(5) *What (efficient) algorithms exist for finding the minimum* p *for which a graph is a* p-d

GIG?

(6) *What (efficient) algorithms exist for coloring* p-d *GIG's?*

We make two conjectures.

Conjecture 6.1: Recognition of a triangulated p-d GIG, $p \in \mathbb{N}$, is a polynomial time problem.

Conjecture 6.2: Suppose G, p, and a are as in Proposition 5.1. Then the following statement is equivalent to the statements given in Proposition 5.1.

(d) In every realization of G^*, $[a]$ has at most one vertex beside it.

Note that G^* is a p-d GIG, since removing all but one vertex from each equivalence class of G yields a graph isomorphic to G^*. The sufficiency of (d) in Conjecture 6.2 follows easily by an argument similar to the proof in Proposition 5.1 (b)$\Rightarrow$(a). The necessity may follow by extending any realization of G^* to a realization of G, after some operations have been carried out on the realization. This may require arguments involving bending and concatenation like those in Lemma 3.5 and Theorem 3.6.

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